JKAU: Sci., **Vol. 21** No. 1, pp: 135-141 (2009 A.D. / 1430 A.H.)

On the Diophantine Equation $x^2 = 4q^n - 4q^m + 9$

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Abstract. In this paper, we study the title equation with q any prime and $n > m \geq 0$, and we give a complete solution when $m > 0$.

Keywords: Diophantine equation, primitive divisor, Lehmer pair.

Introduction

In 1913 the Indian mathematician S. Ramanujan[1] conjectured that the equation $x^2 = 2^n - 7$, had the only following solutions:

$$n = 3 \;\; 4 \;\; 5 \;\; 7 \;\; 15$$

$$x = 1 \;\; 3 \;\; 5 \;\; 11 \;\; 181$$

This conjecture was first proved by Nagell[2] in 1948. There followed during the period 1950-1970, a number of proofs based on a variety of methods (see for example[3,4]).

Ramanujan equation has the general form

$$x^2 = 4q^n - 4q^m + D,$$

where D is any odd integer. The purpose of this paper is to solve

$$x^2 = 4q^n - 4q^m + 9 \qquad (1)$$

with $x > 0$, $n \geq m \geq 0$, where q is any prime, it is clear that x is an odd integer. To solve equation (1) we will use unique factorization of ideals along with linear recurrences and congruences and the BHV Theorem[5].

We start with the case $n = 1$ and $m = 0$.

Case I

Let $(n, m) = (1,0)$, in equation (1), then we have

$$q = \frac{x^2 - 5}{4}.$$

Since x is odd, let $x = 2c + 1$, then we get

$$q = c^2 + c - 1, \tag{2}$$

where c is a positive integer. It is not known if equation (2) has infinitely many solutions.

Case II

Let $d = \text{g.c.d}(m,n)$, $q_1 = q^d$, $n_1 = n/d$, $m_1 = m/d$, in equation (1), then we get the same kind of equation (1)

$$x^2 = 4q_1^{n_1} - 4q_1^{m_1} + 9,$$

with n_1, m_1 are coprime. So we shall suppose $(n, m) = 1$, which means that $n \neq m$.

Case III

If $m = 0$, and n is an even, then equation (1) has no solutions. So we shall exclude all the above cases.

Now we suppose the case $m > 0$, and get the following:

Theorem

The diophantine equation

$$x^2 = 4q^n - 4q^m + 9, \ n > m, \tag{3}$$

has the following two cases:

i. $\boldsymbol{m = 1}$**:** When $q=2$, then it has a unique solution given by $(x,n) = (3,1)$, otherwise it has at most two solutions

ii. $\boldsymbol{m>1}$: **(a)** When m is odd, it has solutions only if $q = 3$, and these solutions are given by

$$(x,n,m) = (93,7,3),\ (2.3^{m-1} - 3,\ 2(m-1),\ m).$$

(b) When m is even, it has solutions only if $q = 2$ and $m = 2$, and these solutions are given by

$$(x,n) = (1,1),\ (3,2),\ (5,3),\ (11,5),\ (181,13).$$

Proof

(i) Let $m = 1$ in (3).

If $q = 2$, then we get equation $x^2 - 1 = 2^{n+2}$, which is clear has a unique solution $(x,q) = (3,2)$. If $q \neq 2$, then the equation $x^2 = 4q^n - 4q + 9$, has at most two solutions[6].

(ii) Let $m > 1$ in (3). We start by writing

$$4q^m - 9 = 4q^n - x^2 = Aa^2. \tag{4}$$

Where $A \geq 1$ is an odd square free. Suppose p divides m, where m is odd and put $q_1 = q^{m/p}$, we get:

$$4q_1^p - 9 = Aa^2 \tag{5}$$

If $A = 1$, then $a^2 \equiv -1 \pmod 4$, but -1 is not quadratic residuce modulo 4, therefore $A \neq 1$.

Now if $A = 3$, then $q = 3$, and dividing equation (4) by 3, we get the equation

$$y^2 = 4q^{n-2} - 4q^{m-2} + 1,$$

which have been solved by Luca[7], and the only solution in our case ($q = 3$) is $n = 7$, $m = 3$ and $y = 31$, so $x = 93$. Also Luca refer to the case $n-2 = 2(m-2)$ as the trivial solution of this equation, and this will give us the solution $x = 2.3^{m-1} - 3$, as desired.

Hence we shall suppose that $q_1 \geq 5$, therefore $A \geq 5$ and $A \not\equiv 0 \pmod 3$. We write (5) as

$$q_1^p = \frac{(3+\sqrt{-A}\,a)}{2}\frac{(3-\sqrt{-A}\,a)}{2}. \qquad (6)$$

Suppose $\langle q_1 \rangle = \pi\ \bar{\pi}$, where π is a prime ideal, therefore the two algebraic integers appearing in the right-hand side of (6) are coprime in the ring $Q(\sqrt{-A})$. Then

$$\pi^p\ \bar{\pi}^p = \left[\frac{3+\sqrt{-A}\,a}{2}\right]\left[\frac{3-\sqrt{-A}\,a}{2}\right].$$

This implies that

$$\pi^p = \left[\frac{3+\sqrt{-A}\,a}{2}\right] \text{ and } \bar{\pi}^p = \left[\frac{3-\sqrt{-A}\,a}{2}\right].$$

So π^p is a principal ideal which implies $\mathrm{O}(\pi)\,|\,p$, hence π is a principal ideal.

Let $z = \frac{c+b\sqrt{-A}}{2}$, where $c \equiv b (\mathrm{mod}\, 2)$, is a generator of π then we get

$$\langle q \rangle = \langle z \rangle \cdot \langle \bar{z} \rangle$$

and

$$\langle z^p \rangle = \left[\frac{3+\sqrt{-A}\,a}{2}\right], \ \langle \bar{z}^p \rangle = \left[\frac{3-\sqrt{-A}\,a}{2}\right].$$

Since the units in the field $Q(\sqrt{-A})$ are ± 1, therefore

$$\pm z^p = \frac{3+\sqrt{-A}\,a}{2}, \quad \pm \bar{z}^p = \frac{3-\sqrt{-A}\,a}{2}.$$

Hence

$$\frac{u_{2p}}{u_p} = z^p + \bar{z}^p = \pm 3 . \qquad (7)$$

From (7) we get that $u_{2p} = \pm 3\, u_p$ which implies that u_{2p} has no primitive divisors.

Let $p=3$ in equation (7), then we get

$$\pm 3 = z^p + \overline{z}^p = \frac{c^3 - 3Acb^2}{4}.$$

Or

$$\pm 12 = c(c^2 - 3Ab^2) \tag{8}$$

If c is even, then so b is even, and in this case the right-hand side of (8) is a multiple of 8, which is impossible. Thus c is an odd divisor of 12, therefore $c = \pm 3, \pm 1$. From equation (8) we now conclude that $3Ab^2 = 5$, $-11, \pm 13$, which is obviously impossible.

Assume now that $p \geq 5$, in this case, u_{2p} has no primitive divisors. From Table 2 and 3 in[5] and a few exceptional values of z. None of the exceptional Lehmer terms from that Table leads to a value of $z \in Q(\sqrt{-A})$. Thus equation (3) has no solutions when m is odd and $q \geq 5$.

Now let us suppose that m is even, say $m = 2k$, and k is a positive integer. From equation (4), we get $x^2 - 9 = 4q^n - 4q^{2k}$, which implies that

$$\frac{x+3}{2} . \frac{x-3}{2} = q^{2k}(q^{n-2k} - 1).$$

Since the two factors in the left hand side are coprime we get $q = 2$ and $m = 2$. Substituting in (4) we find the famous equation of Ramanujan $x^2 = 2^{n+2} - 7$, which has only the following solutions[2]

$$(x,n) = (1,1), (3,2), (5,3), (11,5), (181,13).$$

This concludes the proof.

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دراسة المعادلة الديوفنتية $x^2=4q^n-4q^m+9$

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المستخلص. في هذا البحث درسنا المعادلة الديوفنتية:

$$x^2=4q^n-4q^m+9$$

حيث q عدد أولي و $n > m \geq 0$ أعداد صحيحة و قــدمنا حــلاً كاملاً عندما $m>0$.