Chapter V
Continuous Random Variables
Continuous Probability Distributions

Continuous Random Variables

- Let X be a continuous r. v.

(1) The **probability density function** (PDF) of X is denoted by f (x) and has the properties

(i) f (x) ≥ 0 for all values of x

(ii) $$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

(iii) $$P(a < X < b) = \int_{a}^{b} f(x)\,dx$$

Continuous Random Variables

Example (1):

The PDF of the r. v. X is given by

$$f(x) = \begin{cases} k(4x+3) & 0 < x < 3 \\ zero & otherwise \end{cases}$$

Find the value of k

Solution

$$\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_{0}^{3} k(4x+3)dx = 1 \Rightarrow k = 1/27 = 0.037$$

Continuous Random Variables

- **Example** (2): Find the value of k, given that the PDF of the r. v. X is:

$$f(x) = \begin{cases} k\ x^3 (1-x)^4 & 0 < x < 1 \\ zero & otherwise \end{cases}$$

- **Solution**

$$\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow 1 = \int_0^1 kx^3 (1-x)^4 dx = k \int_0^1 x^3 (1-x)^4 dx = k\ B(4,5)$$

$$k = 1/B(4,5) = 280$$

Continuous Random Variables

- **Example** (3): Find the value of k, given that the PDF of the r. v. X is:

$$f(x) = \begin{cases} k\ x^3 e^{-2x} & x > 0 \\ zero & otherwise \end{cases}$$

- **Solution**

$$k = 1 / \int_0^\infty f(x)dx \qquad \frac{8}{3}$$

Continuous Random Variables

- **Example** (4): Find the value of k, given that the PDF of the continuous r. v. X is:

$$f(x) = \begin{cases} x^2, & 0 < x < 1 \\ k(3x+2), & 1 < x < 3 \\ zero & otherwise \end{cases}$$

- **Solution**

$$1 = \int_0^1 x^2 dx + k \int_1^3 (3x+2) dx \Rightarrow k = 1/24$$

Continuous Random Variables

- **Example** (5): Find the value of k, given that the PDF of the continuous r. v. X is:

$$f(x) = \begin{cases} k(3x^2 + 2x + 5) & 1 < x < 3 \\ zero & otherwise \end{cases}$$

- **Solution**

$$1 = k \int_1^3 (3x^2 + 2x + 5)\, dx = 44k \Rightarrow k = \frac{1}{44}$$

Continuous Random Variables

<u>Example</u> (6): For the PDF of the r. v. X, given by

- We have

$$f(x) = \begin{cases} \frac{1}{27}(4x+3) & 0 < x < 3 \\ zero & otherwise \end{cases}$$

$$P(0 < x < 1) = \int_0^1 \frac{1}{27}(4x+3)\,dx = \frac{5}{27} = 0.185$$

$$P(X > 1.5) = \int_{1.5}^3 \frac{1}{27}(4x+3)\,dx = 0.67$$

Continuous Random Variables

- **<u>Example</u>** (7): Find the value of P(1.5 < X < 2.5), given the PDF of the r. v. X is:

$$f(x) = \begin{cases} \frac{1}{16}(3x+2) \; for \; 1 < x < 3 \\ zero \quad otherwise \end{cases}$$

- **<u>Solution</u>**

$$P(1.5 < X < 2.5) = \int_{1.5}^{2.5} f(x)dx = 0.5$$

Continuous Random Variables

- **Example** (8): Determine P (0.5<X<2.5), given the PDF of the continuous r. v. X

$$f(x) = \begin{cases} x^2, & 0 < x < 1 \\ \frac{1}{24}(3x+2), & 1 < x < 3 \\ zero & otherwise \end{cases}$$

- **Solution**

$$P(0.5 < X < 2.5) = \int_{0.5}^{1} x^2\,dx + \frac{1}{24}\int_{1}^{2.5}(3x+2)\,dx = 0.74$$

Continuous Random Variables

The expected value of X

It is denoted by μ or E(X) and defined by

$$\mu = E(X) \quad \int_{-\infty}^{\infty} x\, f(x)\, dx$$

Higher moments of X

They are defined in a similar way

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x \quad f(x)\, dx, \; r = 0, 1, 2, \ldots$$

Continuous Random Variables

Example (9): Calculate the mean and variance of the r. v. X

$$f(x) = \begin{cases} \frac{1}{27}(4x+3) & 0 < x < 3 \\ zero & otherwise \end{cases}$$

Solution

$$mean = \mu = E(X) = \int_0^3 x \cdot \frac{1}{27}(4x+3)dx = 1.833$$

Continuous Random Variables

- **The variance of X**

$$Var(X) = E(X^2) - {}^2 = \int_0^3 x^2 . \frac{1}{27}(4x+3)dx - \mu^2 = 4 - \left(\frac{11}{6}\right)^2 = 0.639$$

- **The standard deviation of X**

$$\sigma = {}^{+}\sqrt{Var(X)} = {}^{+}\sqrt{0.639} = 0.799$$

Continuous Random Variables

- **Example** (10): Calculate E(X), given the PDF

$$f(x) = \begin{cases} 280x^3(1-x)^4 & 0 < x < 1 \\ zero & otherwise \end{cases}$$

- **Solution**

$$E(X) = \int_0^1 x\,f(x)\,dx = 0.444$$

Continuous Random Variables

- **Example** (11): E(X), given the PDF of X is

$$f(x) = \begin{cases} \frac{8}{3} x^3 e^{-2x} & for\ x > 0 \\ zero & otherwise \end{cases}$$

- **Solution**

$$E(X) = \int_0^{\infty} x\ f(x) dx \qquad 2$$

Continuous Random Variables

- **Example** (12): Calculate E(X), given the PDF

$$f(x)=\begin{cases} x^2, & 0<x<1 \\ \frac{1}{24}(3x+2), & 1<x<3 \\ zero & otherwise \end{cases}$$

- **Solution**

$$E(X)=\int_0^1 x^2\,dx+\int_1^3 x*\frac{1}{24}*(3x+2)\,dx=1.667$$

Continuous Random Variables

- **Example** (13): Calculate $E(X^2)$, given the PDF

$$f(x) = \begin{cases} \frac{1}{16}(3x+2) \; for \; 1 < x < 3 \\ zero \quad otherwise \end{cases}$$

- **Solution**

$$E(X^2) = \int_1^3 x^2 f(x)\,dx = 4.83$$

Continuous Random Variables

- **Example** (14): Calculate E(X^2), given PDF of X is

$$f(x) = \begin{cases} 280\,x^3(1-x)^4 & for \quad 0 < x < 1 \\ zero & otherwise \end{cases}$$

- **Solution**

$$E(X^2) = \int_0^1 x^2 f(x)dx = 0.222$$

Continuous Random Variables

- **Example** (15): Calculate E(X^2), given the PDF

$$f(x)=\begin{cases}\frac{8}{3}x^{3}e^{-2x} & for\ x>0\\ zero & otherwise\end{cases}$$

- **Solution**

$$E(X^{2})=\int_{0}^{\infty}x^{2}f(x)dx \quad 5$$

Continuous Random Variables

- **Example** (16): Calculate $E(X^2)$, given the PDF

$$f(x) = \begin{cases} x^2, & 0 < x < 1 \\ \frac{1}{24}(3x+2), & 1 < x < 3 \\ zero & otherwise \end{cases}$$

- **Solution**

$$E(X^2) = \int_0^1 x^2 x^2 \, dx + \int_1^3 x^2 * \frac{1}{24} * (3x+2) \, dx = 3.422$$

Continuous Random Variables

- **Example** (17): Calculate $E(X^3)$, given the PDF

$$f(x) = \begin{cases} \frac{1}{16}(3x+2) \; for \; 1 < x < 3 \\ zero \quad otherwise \end{cases}$$

- **Solution**

$$E(X^3) = \int_1^3 x^3 f(x)\,dx = 11.575$$

Continuous Random Variables

- **Example** (18): Calculate E(X^3), given the PDF

$$f(x) = \begin{cases} 280x^3(1-x)^4 & for \quad 0 < x < 1 \\ zero & otherwise \end{cases}$$

- **Solution**

$$E(X^3) = \int_0^1 x^3 * f(x)dx = 280\int_0^1 x^6(1-x)^4 dx = 0.121$$

Continuous Random Variables

- **Example** (19): Calculate $E(X^3)$, given the PDF

$$f(x) = \begin{cases} \frac{8}{3} x^3 e^{-2x} & for \ x > 0 \\ zero & otherwise \end{cases}$$

- **Solution**

$$E(X^3) = \int_0^\infty x^3 f(x) dx = \frac{8}{3} \int_0^\infty x^6 e^{-2x} dx = 15$$

Continuous Random Variables

- **Example** (20): Calculate $E(X^3)$, given the PDF

$$f(x) = \begin{cases} x^2, & 0 < x < 1 \\ \frac{1}{24}(3x+2), & 1 < x < 3 \\ zero & otherwise \end{cases}$$

- **Solution**

$$E(X^3) = \int_0^1 x^3 * x^2 \, dx + \int_1^3 x^3 * (3x+2) dx = 7.883$$

Continuous Random Variables

- **Example** (21): Find the CDF F(x) , if the PDF is

$$f(x) = \begin{cases} \frac{1}{16}(3x+2) & 1 < x < 3 \\ zero & otherwise \end{cases}$$

- **Solution**

$$F(x) = \begin{cases} 0 & if \quad x < 1 \\ \frac{3}{32}x^2 + \frac{1}{8}x - \frac{7}{32} & if \quad 1 \le x < 3 \\ 1 & if \quad x \ge 3 \end{cases}$$

Continuous Random Variables

Example (22): Find the CDF F(x) if the PDF is

$$f(x)=\begin{cases} x^2, & 0<x<1 \\ \frac{1}{24}(3x+2), & 1<x<3 \\ zero & otherwise \end{cases}$$

Solution

$$F(x)=\begin{cases} 0 & x<0 \\ \frac{x^3}{3}, & 0\le x<1 \\ \frac{3}{16}x^2+\frac{1}{12}x-\frac{7}{48}, & 1\le x<3 \\ 1 & 3\le x \end{cases}$$

Continuous Random Variables

1. **Continuous uniform distribution on [a, b]**

- The PDF of X is given by

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & otherwise \end{cases}$$

- In this case, we write X ~ U[a, b]
- Mean $E(X) = \dfrac{a+b}{2}$
- Variance $Var(X) = \dfrac{(b-a^{\;2}}{12}$

Continuous Random Variables

- The CDF

$$F(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x < b \\ 1 & b \le x \end{cases}$$

- The MGF

$$M_X(t) = \frac{e^{bt} - e^{at}}{(b-a)\mathrm{t}}$$

Continuous Random Variables

Example (23): The continuous r. v. X has a uniform distribution on the interval [a, b] = [– 1, 1]. Write down the PDF of X.

Solution

$$f(x) = \begin{cases} \frac{1}{2} & -1 \le x \le 1 \\ 0 & otherwise \end{cases}$$

Continuous Random Variables

Example (24): The continuous r. v. X has a uniform distribution on the interval [a, b] = [– 1, 1]. What is the expected value of X?

Solution

$$E(\mathrm{X}) = \frac{1+(-1)}{2} = 0$$

Continuous Random Variables

- **Example** (25): The continuous r. v. X has a uniform distribution on the interval [a, b] = [– 1, 1]. What is the variance of X?
- **Solution**

$$Var(\mathrm{X}) = \frac{(1-(-1))^2}{12} = 0.333$$

Continuous Random Variables

Example (26): The continuous r. v. X has a uniform distribution on the interval [a, b] = [– 1, 1]. Write down the CDF.

Solution

$$F(x) = \begin{cases} 0 & x < -1 \\ \dfrac{x+1}{2} & -1 \le x < 1 \\ 1 & 1 \le x \end{cases}$$

Continuous Random Variables

Example (27): The continuous r. v. X has a uniform distribution on the interval [a, b] = [– 1, 1]. Find the MGF of X .

Solution

$$M_X(\ ) = \frac{e^{t} - e^{-t}}{2t}$$

Continuous Random Variables

The exponential distribution

The PDF of X is given by

$$f(x) = \begin{cases} e^{-\lambda x} & x \geq 0, \lambda \quad 0 \\ zero & otherwise \end{cases}$$

In this case, we write X ~ Exp (λ).

- Mean $E(X) = \frac{1}{\lambda}$

Continuous Random Variables

- The variance

$$Var(X) = \frac{1}{\lambda^2}$$

- The CDF

$$F(x) = \begin{cases} 1 \quad e^{-\lambda x} & for \ x \geq \\ zero & for \ x < 0 \end{cases}$$

- The MGF

$$M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-1}, \ t < \lambda$$

Continuous Random Variables

- **Example** (28): Find the CDF F(x) , if the PDF is

$$f(x) = \begin{cases} 5e^{-5x} & for\ x \geq 0 \\ zero & otherwise \end{cases}$$

- **Solution**

$$F(x) = \begin{cases} 1-e^{-5x} & for\ x \geq \\ zero & for\ x < 0 \end{cases}$$

Continuous Random Variables

- **Example** (29): Find the MDF $M_X(t)$, if the PDF is

$$f(x)=\begin{cases} 5e^{-5x} & for\ x \geq 0 \\ zero & otherwise \end{cases}$$

- **Solution**

$$M_X(t)=\left(1-\frac{t}{5}\right)^{-1},\ t<5$$

Continuous Random Variables

- **<u>Example</u>** (30): Find the expected value E(X), if the probability density function is given by:

$$f(x) = \begin{cases} 5e^{-5x} & for\ x \geq 0 \\ zero & otherwise \end{cases}$$

- **<u>Solution</u>**

$$E(X) = \frac{1}{\lambda} = \frac{1}{5} = 0.2$$

Continuous Random Variables

<u>Example</u> (31): Find the value of Var (X), if the PDF is given by:

$$f(x) = \begin{cases} 5e^{-5x} & for\ x \geq 0 \\ zero & otherwise \end{cases}$$

<u>Solution</u>

$$Var(X) = \frac{1}{25}$$

Continuous Random Variables

3. Beta distribution

It has a PDF of the form

$$f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x) \quad , 0 < x < 1$$

In this case, we write X ~ Beta (a, b).

Continuous Random Variables

Example (32): Find the value of E (X^r)) for the PDF

$$f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}, 0 < x < 1$$

Solution

$$E(X^{r}) \quad \frac{B(a+r,b)}{B(a,b)}, r = 0,1,2,3,\ldots$$

Continuous Random Variables

Example (33): Find E (X) given that the r-th moment $E(X^r)$ of the r. v. X having Beta (a, b) = Beta (3, 5) distribution is:

$$E(X^{r}) \quad \frac{B(3+\quad,5)}{B(3,5)}, \; r=0,1,2,\ldots$$

- **Solution**

$$E(X)=\frac{3}{3+5}=0.375$$

Continuous Random Variables

Example (34): Find E (X^2) given that the r-th moment E (X^r) of the r. v. X having Beta (a, b) = Beta (3, 5) distribution is:

$$E(X^r) \quad \frac{B(3+\quad,5)}{B(3,5)}, r = 0,1,2,\ldots$$

Solution

$$E(X^2) = \frac{B(3+2,5)}{B(3,5)} = \frac{(5,5)}{B(3,5)} = \frac{12}{72} = 0.167$$

Continuous Random Variables

Example (35): Find Var (X) given the r-th moment E (X^r) of the r. v. X having Beta (a, b) = Beta (3, 5) distribution is:

$$E(X^r) \quad \frac{B(3+r,5)}{B(3,5)}, r = 0,1,2,...$$

- **Solution**

$$Var(X) = \frac{B(5,5)}{B(3,5)} - \left[\frac{(4,5)}{B(3,5)}\right]^2 = \frac{15}{576} = 0.026$$

Continuous Random Variables

3. **Gamma distribution**

It has a PDF of the form

$$f(x)=\begin{cases}\dfrac{1}{\beta^{\alpha}\,\Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta} & x>0,\ \alpha>0,\ \beta>0\\ zero & otherwise\end{cases}$$

In this case, we write X ~ Gamma (α, β). For λ=1/β

$$f(x)=\begin{cases}\dfrac{\lambda^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x} & x>0,\ \alpha>0,\ \lambda>0\\ zero & otherwise\end{cases}$$

Continuous Random Variables

- Mean $E(X) = \frac{\alpha}{\lambda}$
- Variance $Var(X) = \frac{\alpha}{\lambda^2}$
- The MGF $M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-\alpha}, t < \lambda$

Continuous Random Variables

Example (36): Let X~ gamma (α, λ) = gamma (2, 3).

Obtain the value of E (X) and Var (X).

Solution

$$E(X) = \frac{2}{3} = 0.667$$

$$Var(X) = \frac{2}{9} = 0.222$$

Continuous Random Variables

Example (39): Let X ~ gamma (α, λ) = gamma (2, 3).

Find the moment generating function $M_X(t)$.

Solution

$$M_X(t) = \left(1 - \frac{t}{3}\right)^{-2}, t < 3$$

Continuous Random Variables

- **Example** (40):The MGF of the random variable X is given by

$$M_X(t) = \left(1 - \frac{t}{4}\right)^{-3}, t < 4$$

Find E(X), E(X^2), and Var (X).

- **Solution**

E(X) = ¾ = 0.75

E(X^2) = 3/4 = 0.75

Var (X) = 0.75 – $(0.75)^2$ = 0.188

Continuous Random Variables

- **Example** (41): The MGF of the r. v. X is

$$M_X(t) = (1-3t)^{-1}, t < 1/3$$

Find E(X), $E(X^2)$, and Var (X).

- **Solution**

E(X) = 3

$E(X^2) = 18$

Var (X) = $18 - 3^2 = 9$

Continuous Random Variables

<u>Normal distribution</u>

If the PDF of the r. v. X has the form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{1}{2\sigma^2}(x-\mu)^2}, -\infty < x < +\infty, -\infty < \mu < +\infty, 0 < \sigma < \infty$$

we say that the r. v. X has a normal distribution with parameters μ and σ. This is denoted by

$$X \sim N(\mu, \sigma^2)$$

Continuous Random Variables

Characteristics of the normal distribution

- The curve $y = f(x)$ is a bell-shaped curve and is symmetric about the line $x = \mu$.
- The area under the curve $y = f(x)$ from $-\infty$ to $+\infty$ equals 1.
- The curve $y = f(x)$ has two points of inflection at $x = \mu \pm \sigma$
- The area under the normal curve is given by

Continuous Random Variables

The 3 – sigma rule. Namely,

(i) The area from $\mu - \sigma$ to $\mu + \sigma$ is 0.68

(ii) The area from $\mu - 2\sigma$ to $\mu + 2\sigma$ is 0.95

(iii) The area from $\mu - 3\sigma$ to $\mu + 3\sigma$ is 0.997

- The MGF of the normal distribution $N(\mu, \sigma^2)$ has the form

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

Continuous Random Variables

- **Example** (42): Let $X \sim N(\mu, \sigma^2) = N(100, 25)$. Between what two values lies 0.68, 0.95, and 0.997 of the area under the normal curve.
- **Solution**
- 0.68 of area lies between $(\mu - \sigma , \mu + \sigma) = (95, 105)$
- 0.95 of area lies between $(\mu - 2\sigma , \mu + 2\sigma) = (90, 110)$
- 0.997 of area lies between $(\mu - 3\sigma, \mu + 3\sigma) = (85, 115)$

Continuous Random Variables

Example (43): Let X ~ N (μ, σ^2) = N (100, 50). Find the MGF of X.

Solution

$$M_X(t) = e^{\mu t + 1/2\sigma^2 t^2}$$

Therefore

$$M_X(t) = e^{100t + 25t^2}$$

Continuous Random Variables

- **Remark**:
- Let X ~ N(μ , σ²). Let Z = (X - μ) / σ. The random variable Z will be normally distributed with mean 0 and variance 1. That is Z ~ N(0, 1)
- N(0, 1) distribution is called the standard normal distribution. The PDF of N(0, 1) is

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, -\infty < z < +\infty$$

Continuous Random Variables

Characteristics of standard normal distribution

- The curve $y = \phi(x)$ is a bell-shaped curve and is symmetric about the line $z = 0$.
- The area under the curve $y = \phi(x)$ from $-\infty$ to $+\infty$ equals 1.
- The curve $y = \phi(x)$ has two points of inflection at $z = \pm 1$
- The area under the standard normal curve is given by

Continuous Random Variables

The 3 – sigma rule. Namely,

(i) The area from – 1 to + 1 is 0.68

(ii) The area from - 2 to + 2 is 0.95

(iii) The area from - 3 to + 3 is 0.997

- The MGF of the normal distribution N(0 , 1) has the form

$$M_Z(t) = e^{\frac{1}{2}t^2}$$

Continuous Random Variables

- The cumulative distribution function of the standard normal distribution is

$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

and its values are tabulated.

Continuous Random Variables

- **Example** (21): Let $X \sim N(100, 25)$.

(i) $P(X < 110) = P(Z < 2) = \Phi(2) = \ldots$

(ii) $P(X > 115) = P(Z > 3) = 1 - \Phi(3) = \ldots$

(iii) $P(105 < X < 115) = P(1 < Z < 3)$

$$= \Phi(3) - \Phi(1) = \ldots$$

(i) $P(X < 97) = P(Z < -0.6) = \Phi(-0.6)$

$$= 1 - \Phi(0.6) = \ldots$$

$\Phi(\ )$ is tabulated in the normal table.