CHAPTER III
CONDITIONAL PROBABILITY

CONDITIONAL PROBABILITY

- The probability that event A, given that event B occurred is called the conditional probability of A given B and denoted by P(A|B)

$$P(A\,|\,B)=\frac{P(A\cap B)}{P(B)},\,P(B)>0$$

- Note that the occurrence of event B precedes the occurrence of event A.

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- The conditional probability of B, given A is

$$P(B \mid A) = \frac{P(A \cap B}{P(A)}, P(A) > 0$$

- It follows that

$$P(A \cap B) = P(A)P(B \mid A) = P(B)P(A \mid B)$$

- This is called the multiplication rule in probability

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- Multiplication Rule for 3 events states

$$P(A \cap B \cap C) = P(A)P(B \mid A)P(C \mid A \cap B)$$

- In general, we have

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2)\ldots P\left(A_n \mid \bigcap_{i=1}^{n-1} A_i\right)$$

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Example (1) Let A, B be defined on S such that:

$$P(A) = 0.38, P(B) = 0.45, P(A \cup B) = 0.65$$

Find P(A|B), P(B|A).

Solution:

$P(A \cap B) = 0.38 + 0.45 - 0.65 = 0.18$

$P(A \mid B) = 0.18 / 0.45 = 0.40$

$P(B \mid A) = 0.18 / 0.38 = 0.47$

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Example (2)

Two cards are drawn at random and in succession from an ordinary deck of 52 playing cards. Find the probability that both cards will be Hearts, if the drawing was:

(1) with replacement

(2) without replacement

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Solution:

Let H_i = event i th card is Heart, i = 1, 2.

Required Probability = $P(H_1 \cap H_2)$

$= P(H_1)\, P(H_2 \mid H_1)$

(1) Req. Prob. = (13/52) x (13/52) = 0.063

(2) Req. Prob. = (13/52) x (12/51) = 0.059

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Example (3)

Three cards are drawn at random and in succession from an ordinary deck of 52 playing cards. Find the probability that all cards will be Hearts, if the drawing was:

(1) with replacement

(2) without replacement

CONDITIONAL PROBABILITY

Solution:

Let H_i = event i th card is Heart, i = 1, 2, 3.

Required Probability = $P(H_1 \cap H_2 \cap H_3)$

$= P(H_1)\, P(H_2 \mid H_1)\, P(H_3 \mid H_1 \cap H_2)$

(1) Req. Prob. = $(13/52)^3 = 0.016$

(2) Req. Prob. = (13/52) x (12/51) x (11/50)

= 0.013

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Independence of events

Two events A &B are said to be independent if

$$P(A|B) = P(A)$$

Or, equivalently

$$P(B|A) = P(B)$$

Consequently, for independent events:

$$P(A \cap B) = P(A)\, P(B)$$

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Example (4)

A large city has two fire-engines operating independently. The probability that a specific engine is available when needed is 0.85. Find:

(1) P(an engine is available when needed)

(2) P(neither engine is available when needed)

Solution: Let E_i be the event " i^{th} engine is available when needed", i = 1, 2.

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$P(E_1) = P(E_2) = 0.85,$

$P(E_1 \cap E_2) = P(E_1)P(E_2) = 0.85 \text{ x } 0.85 = 0.7225$

(1) Req. Prob. $= P(E_1 \cup E_2)$

$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$= 0.85 + 0.85 - 0.7225 = 0.9775$

(2) Req. Prob. $= P[(E_1 \cup E_2)^c] = 1 - P(E_1 \cup E_2)$

$= 1 - 0.9775 = 0.0225$

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A, B, and C are said to be independent if:

(1) $P(A \cap B) = P(A)\,P(B)$

(2) $P(A \cap C) = P(A)\,P(C)$

(3) $P(B \cap C) = P(B)\,P(C)$

(4) $P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$.

Remark: If conditions (1) – (3) are satisfied, we say that A, B, and C are pairwise independent

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Example (5)

A large city has 3 fire-engines operating independently. The probability that a specific engine is available when needed is 0.85. Find:

(1) P(an engine is available when needed)

(2) P(neither engine is available when needed)

Solution: Let E_i be the event " i^{th} engine is available when needed", i = 1, 2, 3.

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$P(E_1) = P(E_2) = P(E_3) = 0.85,$

$P(E_1 \cap E_2) = P(E_1)P(E_2) = 0.85 \times 0.85 = 0.7225$

$P(E_1 \cap E_3) = P(E_1)P(E_3) = 0.85 \times 0.85 = 0.7225$

$P(E_2 \cap E_3) = P(E_2)P(E_3) = 0.85 \times 0.85 = 0.7225$

$P(E_1 \cap E_2 \cap E_3) = P(E_1)P(E_2)P(E_3)$

$= 0.85 \times 0.85 \times 0.85$

$= 0.6141$

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(1) Req. Prob. = $P(E_1 \cup E_2 \cup E_3)$

$= 3 \times (0.85) - 3 \times (0.7225) + 0.6141$

$= 0.9966$

(2) Req. Prob. = $P[(E_1 \cup E_2 \cup E_3)^c]$

$= 1 - P(E_1 \cup E_2 \cup E_3)$

$= 1 - 0.9966 = 0.0034$

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Partition of the Sample Space

The collection A_1 , A_2 , …, A_n is said to be a partition of S if:

(1) They are mutually exclusive

$$A_i \cap A_j = \Phi, \forall\, i \neq j$$

(2) Their union equals S.

$$A_1 \cup A_2 \cup \ldots \cup A_n = S$$

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The Law of Total Probability

Let A_1 , A_2 , …, A_n be a partition of S. Let D be an event defined on S. Then

$$P(D) = \sum_{i=1}^{n} P(A_i)P(D \mid A_i)$$

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Bayes' Rule

Let A_1 , A_2 , ..., A_n be a partition of S. Let D be an event, defined on S. Then

$$P(A_k \mid D) = \frac{P(D \mid A_k)P(A_k)}{\sum_{i=1}^{n} P(A_i)P(D \mid A_i)}, k = 1, 2, \ldots, n$$

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Example (6) The distribution of colored balls in two boxes is as follows:

Box	Red	White	Black
I	4	6	5
II	7	8	4

A ball is selected at random from Box I and put unseen into Box II. Then, a ball is selected at random from Box II.

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It is required to find:

1. P(both selected balls have the same color)

Solution:

$$\text{Req. Prob.} = P(W_1 \cap W_2) + P(R_1 \cap R_2) + P(B_1 \cap B_2)$$

$$= P(W_1)\, P(W_2|W_1) + P(R_1)\, P(R_2|R_1) + P(B_1)\, P(B_2|B_1)$$

$$= 6/15 \times 9/20 + 4/15 \times 8/20 + 5/15 \times 5/20$$

$$= 0.37$$

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2. P(both selected balls have different colors)

$= 1 - P(\text{both have the same color}) = 0.63.$

3. P(the second ball is white) = $P(W_2)$

$= P(W_2 \mid W_1)P(W_1) + P(W_2 \mid R_1)P(R_1) + P(W_2 \mid B_1)P(B_1)$

$= 9/20 \times 6/15 + 8/20 \times 4/15 + 8/20 \times 5/15$

$= 126/300 = 0.42$

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3. P(first ball is White, given the second was White) =

$$P(W_1|W_2) = \frac{P(W_2|W_1)P(W_1)}{P(W_2)} = \frac{(9/20)(6/15)}{0.42} = 0.429$$

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Example (7) Three machines produce respectively 0.35, 0.37, and 0.28 of the total production of a given item at a certain factory. The probabilities of producing a defective item on these machines are 0.07, 0.05, and 0.08 respectively. An item is selected at random. Find the probability that the selected item is defective.

Solution

Req. = 0.35 x 0.07 + 0.37 x 0.05 + 0.28 x 0.08 = 0.0654

CONDITIONAL PROBABILITY

Example (8) Three machines produce respectively 0.35, 0.37, and 0.28 of the total production of a given item in a certain factory. The probabilities of producing a defective item on these machines are 0.07, 0.05, and 0.08 respectively. An item is selected at random and found defective. Find the probability that the selected defective item is produced by the second machine.

- Req. = [0.37 x 0.05] / [0.35 x 0.07 + 0.37 x 0.05 + 0.28 x 0.08] = 0.283

CONDITIONAL PROBABILITY

Example (9) Three machines produce respectively 0.35, 0.37, and 0.28 of the total production of a given item in a certain factory. The probabilities of producing a defective item on these machines are 0.07, 0.05, and 0.08 respectively. An item is selected at random and found non-defective. Find the probability that the selected non-defective item is produced by the second machine.

- Req. = $[0.37 \times \{1 - 0.05\}] / [1 - \{0.35 \times 0.07 + 0.37 \times 0.05 + 0.28 \times 0.08\}] = 0.376$

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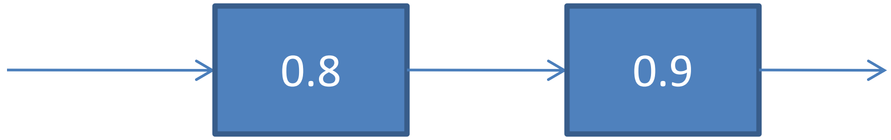
Example (10) The following circuit operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph.



Assume that devices fail independently. What is the probability that the circuit operates?

- Required = 0.8 x 0.9 = 0.72

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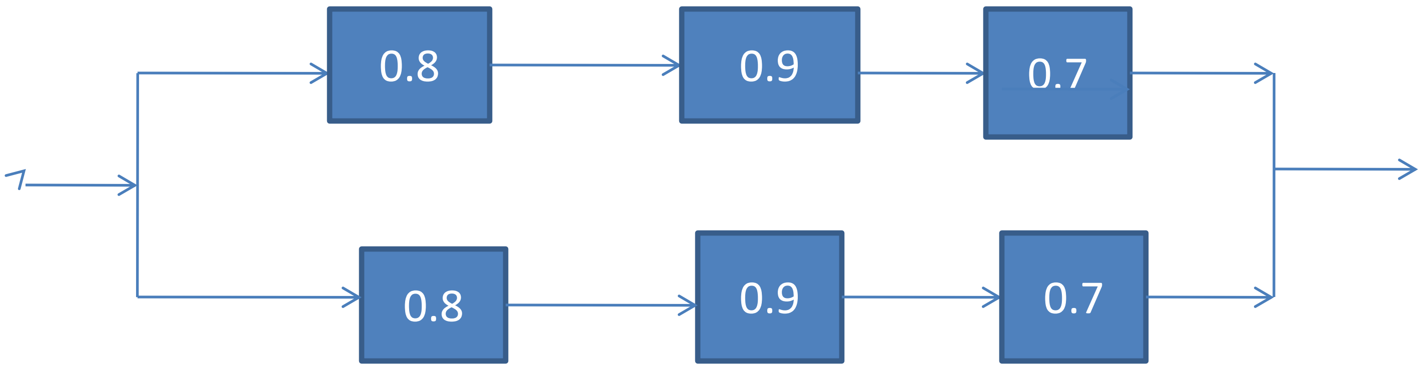
Example (11) consider the following system



Assume that devices fail independently. What is the probability that the circuit operates?

- Required = 0.754