KING ABDULAZIZ UNIVERSITY
FACULTY OF SCIENCE
DEPARTMENT OF STATISTICS

STAT 210
THEORY OF PROBABILITY

COMPUTER SCIENCE STUDENTS

CHAPTER I
COMBINATORIAL ANALYSIS

COMBINATORIAL ANALYSIS

- **Basic Counting Principle**:

  If a process P_1 can occur in n_1 different ways

  and to each of these ways

  a process P_2 can occur in n_2 different ways,

  then

  P_1 and P_2 will occur in $n_1 \times n_2$ different ways

COMBINATORIAL ANALYSIS

- **Example (1)**:

Process 1: going from Home to Transport station

Let P_1 can occur in $n_1 = 3$ ways.

Process 2: going from transport station to work

Let P_2 can occur in $n_2 = 4$ ways.

P_1 and P_2 can occur in $n_1 \times n_2 = 3 \times 4 = 12$ ways.

COMBINATORIAL ANALYSIS

- **<u>Generalized Counting Principle</u>**

  If a process P_1 can occur in n_1 different ways and to each of these ways a process P_2 can occur in n_2 different ways, and so on ..., then Processes P_1 , P_2 , ..., and P_k will occur in

$$n_1 \times n_2 \times \ldots \times n_k$$

  different ways.

COMBINATORIAL ANALYSIS

- **<u>Example</u>** <u>(2)</u>:

  From a group of 10 American, 6 French, and 8 Russian of multi-national peace forces, a subgroup of 3 persons (1 from each nationality) is to be selected. How many such subcommittees can be formed?

  <u>Solution</u>:

  Required Number = 10 x 6 x 8

  = 480 subcommittee

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- **Permutations**:
- A permutation of n different objects is an arrangement of these n objects.
- The number of permutations of n different objects, taken all at a time, is n!
- The number of permutations of **n** different objects, taken **r** ($r \le n$) at a time, is denoted and defined by

$$ {}^{n}P_{r} = \frac{n!}{(n-r)!}, \; 0 \le r \le n $$

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- **Remarks**:

  (i) ${}^{n}P_{0} = 1$

  (ii) ${}^{n}P_{n} = n!$

  (iii) ${}^{n}P_{1} = n$

  (iv) ${}^{n}P_{r} = n\,(n - 1)\,(n - 2) \ldots (n - r + 1)$

COMBINATORIAL ANALYSIS

- **Example** (3):

  A group of 6 American, 5 French, and 4 Russian of multi-national peace forces, is to be ordered in a row. Determine the number of possible orderings if no additional restrictions are imposed.

  Solution:

  Required Number = $(6 + 5 + 4)! = 15!$

COMBINATORIAL ANALYSIS

- **Example** (4):

A group of 6 American, 5 French, and 4 Russian of multi-national peace forces, is to be ordered in a row. Find the number of possible American – French – Russian orderings, if each nationality is to be ordered within itself.

Solution:

Required Number = 6! X 5! X 4! = 2073600

COMBINATORIAL ANALYSIS

- **Example** (5):

A group of 6 American, 5 French, and 4 Russian of multi-national peace forces, is to be ordered in a row. Find the number of possible orderings in any order, if each nationality is to be ordered within itself.

Solution:

$$\text{Required Number} = 3! \times (6! \times 5! \times 4!)$$
$$= 12441600$$

COMBINATORIAL ANALYSIS

- **<u>Example</u>** <u>(6)</u>:

  By rearrangement of the letters of the word "PETROLIUM ", the number of

(i) 2-letter words = $^9P_2 = 9 \times 8 = 72$

(ii) 3-letter words = $^9P_3 = 9 \times 8 \times 7 = 504$

(iii) 4-letter words = $^9P_4 = 9 \times 8 \times 7 \times 6 = 3024$

(iv) 5-letter words = $^9P_5 = 9 \times 8 \times 7 \times 6 \times 5 = 15120$

COMBINATORIAL ANALYSIS

(v) 6-letter words = ${}^9P_6$

$= 9 \times 8 \times 7 \times 6 \times 5 \times 4 = 60480$

(vi) 7-letter words = ${}^9P_7$

$= 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 = 181440$

(vii) 8-letter words = ${}^9P_8$

$= 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 362880$

(viii) 9 – letter words = 9! = 362880

COMBINATORIAL ANALYSIS

- **Combinations :**

  The number of different groups of **r** objects, that could be formed of a total of **n** objects is denoted and defined by

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!(n-r)!}, r \leq n$$

COMBINATORIAL ANALYSIS

- **Remarks**:

  (i) $^{n}C_{0} = 1$

  (ii) $^{n}C_{1} = n$

  (iii) $^{n}C_{n} = 1$

  (iv) $^{n}C_{r} = {}^{n}C_{n-r}$

COMBINATORIAL ANALYSIS

- **<u>Example (6)</u>:**

  When tossing a coin 4 times, the number of

(i) outcomes with no Heads = ${}^4C_0 = 1$

(ii) outcomes with 1Head = ${}^4C_1 = 4$

(iii) outcomes with 2 Heads = ${}^4C_2 = 6$

(iv) outcomes with 3 Heads = ${}^4C_3 = 4$

(v) outcomes with 4 Heads = ${}^4C_4 = 1$

COMBINATORIAL ANALYSIS

- **Example (7)**:

  Five persons are selected at random from a group of 6 men and 8 women. Find the number of selections with 3 men & 2 women.

- **Solution**:

  Required Number = ${}^{6}C_{3} \times {}^{8}C_{2} = 20 \times 28 = 560$

Remark : The selection of persons is logically without replacement.

COMBINATORIAL ANALYSIS

- **Example (8)**:

  Seven persons are selected at random from a group of 6 men, 8 women, and 4 children. Find the number of selections with 3 men, 2 women and 2 children.

- **Solution**:

  Required Number = ${}^{6}C_{3} \times {}^{8}C_{2} \times {}^{4}C_{2}$

  $= 20 \times 28 \times 6 = 3360$

COMBINATORIAL ANALYSIS

- **Example (9)**:

  Four persons are selected at random from a group of 6 men, 8 women, and 4 children. Find the number of selections with equal number of men and women

- **Solution**:

  $$\text{Req.} = {}^{6}C_{0} \times {}^{8}C_{0} \times {}^{4}C_{4} + {}^{6}C_{1} \times {}^{8}C_{1} \times {}^{4}C_{2} + {}^{6}C_{2} \times {}^{8}C_{2} \times {}^{4}C_{0} = 709$$

COMBINATORIAL ANALYSIS

- **Example (10)**:

  Five persons are selected at random from a group of 6 men, 8 women, and 4 children. Find the number of selections with 2 men and at least 1 woman.

- **Solution**:

  $$\text{Required} = {}^{6}C_{2} \times {}^{8}C_{1} \times {}^{4}C_{2} + {}^{6}C_{2} \times {}^{8}C_{2} \times {}^{4}C_{1} + {}^{6}C_{2} \times {}^{8}C_{3} \times {}^{4}C_{0} = 3240$$

COMBINATORIAL ANALYSIS

- **Example (11)**:

  Four persons are selected at random from a group of 6 men, 8 women, and 4 children. Find the number of selections in which all persons have the same gender

- **Solution**:

  $$\text{Required Number} = {}^{6}C_{4} + {}^{8}C_{4} + {}^{4}C_{4}$$

  $$= 86$$

COMBINATORIAL ANALYSIS

- **Example (12)**:

  Four persons are selected at random from a group of 6 men, 8 women, and 4 children. Find the number of selections in which "all selected persons are not of the same gender".

- **Solution**:

  Required = N(S) – N(all of the same gender)

  $$= {}^{18}C_4 - [\,{}^{6}C_4 + {}^{8}C_4 + {}^{4}C_4\,]$$

  $$= 3060 - 86 = 2974$$

COMBINATORIAL ANALYSIS

- **Example (13)**:

  Four persons are selected at random from a group of 6 men, 8 women, and 4 children. Find the number of selections in which "all genders are present" in the selected sample.

- **Solution**:

  Required $= {}^{6}C_{2}\,{}^{8}C_{1}\,{}^{4}C_{1} + {}^{6}C_{1}\,{}^{8}C_{2}\,{}^{4}C_{1} + {}^{6}C_{1}\,{}^{8}C_{1}\,{}^{4}C_{2}$

  $= 480 + 672 + 288$

  $= 1440$

COMBINATORIAL ANALYSIS

Binomial Expansion

(n positive integer)

$$(x+y)^n = \sum_{k=0}^{n} {}^{n}C_k \, y^k x^{n-k} = {}^{n}C_0 x^n y^0 + {}^{n}C_1 x^{n-1} y^1 + \ldots + {}^{n}C_n x^0 y^n$$

$$(x+y)^n = x^n + n\,x^{n-1} y^1 + \frac{n(n-1)}{2!} x^{n\;\;2} y^2 + \ldots + y^n$$

COMBINATORIAL ANALYSIS

<u>Example</u> (14):

$$(x+y)^4 = \sum_{k=0}^{4} {}^4C_k\, y^k x^{4-} = {}^4C_0 x^4 + {}^4C_1 x^3 y^1 + \ldots + {}^4C_4 y^4$$

$$(x+y)^4 = x^4 + 4x^3y^1 + 6x^2y^2 + 4x\ y^3 + y^4$$

The expansion contains 5 terms

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- The coefficients in the binomial expansion can be determined from Pascal's triangle:

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

COMBINATORIAL ANALYSIS

- **Example (15)**:
- In the expansion of $(x + y)^5$ the coefficients are respectively: 1, 5, 10, 10, 5, 1.

$$(x+y)^5 = x^5 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5x\,y^4 + y^5$$

COMBINATORIAL ANALYSIS

- **Multinomial coefficients**:

The number of permutations of n objects that contains k_1 of type 1, k_2 of type 2, … , k_r of type r is denoted and defined by:

$$\binom{n}{k_1, k_2, \ldots, k_r} = \frac{n!}{k_1! k_2! \ldots k_r!}, \quad k_1 + k_2 + \ldots + k_r = n$$

COMBINATORIAL ANALYSIS

- **Example (16)**:

  Determine the number of words that can be formed by rearranging the letters of the word "STATISTICS".

  Solution. We have 10 letters containing : 3 "S", 3 "T", 2 "I", 1 "A", and 1 "C".

  Required Number = 10! / [3! x 3! x 2! x 1! x 1!]

  = 50400