CHAPTER 4
SOME DISCRETE PROBABILITY DISTRIBUTIONS

Discrete Probability Distributions

1. **Bernoulli distribution**

Definition "Bernoulli trial"

It is a random experiment, whose outcomes is either:

(a) "**A = success**" with probability p, $0<p<1$, or

(b) "**not A = failure**" with probability $1 - p = q$

Examples :

(i) Tossing a coin once and looking for "Head"

(ii) Rolling a die once and looking for "face 1"

Discrete Probability Distributions

Now let X denote the number of successes in one Bernoulli trial.

The possible values of X are 0, 1.

It is clear that X is a discrete r. v. with

$P(X = 1) = P(\text{success}) = p, 0 < p < 1$

$P(X = 0) = P(\text{failure}) = q, 0 < q < 1$

Such that $p + q = 1$.

Discrete Probability Distributions

The probability mass function (PMF) of X is

$$f(x) = P(X = x) = p^x q^{1-x}, \ x = 0, 1, \ 0 < p, q < 1, \ p + q = 1$$

This is called the Bernoulli distribution with one parameter p.

This is referred to simply, as

$$X \sim Ber(p)$$

Discrete Probability Distributions

Characteristics of Bernoulli distribution

(a) Mean E(X) = p

(b) Variance Var(X) = p q

(c) Standard deviation $\sigma = \sqrt{p\,q}$

(d) The moment generating function

$$M_X(t) = E[e^{tX}] = q + p\,e^t$$

(e) The probability generating function is

$$G_X(t) = E[t^X] = q + p\,t$$

Discrete Probability Distributions

2. **Binomial distribution**

Consider n Bernoulli trials such that:

C1. The outcome of each trial is either "success" with probability p, or "failure" with probability q

C2. The probability p of success remains constant, i.e. it doesn't change from trial to trial.

C3. The trials are independent.

Discrete Probability Distributions

Let X be a r. v. denoting the number of successes in these n trials

The possible values of X are X = 0, 1, 2, ..., n

It is clear that X is a discrete r. v.

The probability mass function of X is

$$f(x) = P(X = x) = \binom{n}{x} p^x q^{n-x}, \; x = 0, 1, 2, \ldots, n$$

This is called binomial distribution with parameters n and p. We write $X \sim Bin(n, p)$

Discrete Probability Distributions

Characteristics of binomial distribution

(a) Mean = E(X) = n p

(b) Variance = Var(X) = n p q

(c) Standard deviation $\sigma = \sqrt{n\,p\,q}$

(d) Moment generating function

$$M_X(t) = E[e^{tX}] = \left(q + p\,e^{t}\right)^{n}$$

(e) Probability generating function $G(t) = \left(q + p\,t\right)^{n}$

Discrete Probability Distributions

Example (1):

A fair coin is tossed 6 times. Find the probability that the Head will appear:

(a) Exactly 3 times

(b) At least 2 times

(c) At most 4 times

Solution

Discrete Probability Distributions

We have

$$f(x) = P(X = x) = \binom{6}{x}\left(\frac{1}{2}\right)^{6}, x = 0, 1, \ldots, 6$$

(a) P(X=3) = f(3)=0.313

(b) P(X ≥ 2) = f(2) + f(3) + … + f(6) = 1 – [f(0)+f(1)]

= 1 – 0.109 = 0.891

(c) P(X ≤ 4) = 1 – [f(5) + f(6)] = 1 – 0.109 = 0.891

Discrete Probability Distributions

- **<u>Example (2)</u>** :

A person fires at a certain target in 6 independent successive trials with a constant probability 0.9 of hitting the target. Find the probability of hitting the target:

(i) Exactly 2 times

(ii) At least 4 times

(iii) At most 3 times

Discrete Probability Distributions

- **Solution** :

  Let X = number of times of hitting the target.

  X ~ Bin (6, 0.9).

$$f(x) = P(X = x) = \binom{6}{x} (0.9)^x \; (0.1)^{6-x} \; , x = 0, 1, 2, \ldots, 6$$

(i) Req. Prob. = P(X = 2) = f(2) = …

(ii) Req. Prob. = P(X ≥ 4) = f(4) + f(5) + f(6) = …

(iii) Req. Prob. = P(X≤ 3) = f(0)+f(1)+ f(2)+f(3)= …

Discrete Probability Distributions

- **Example (3)** :

  Find the mean, the variance, and the standard deviation of the number of hitting in **Example (2)**.

- Solution

Mean = E(X) = n p = 6 * 0.9 = 5.4

Variance = Var (X) = n p q = 6* 0.9 * 0.1 = 0.54

Standard deviation = σ = 0.73

Discrete Probability Distributions

Example (4): The moment generating function of the r. v. X is

$$M_X(t) = \left(0.25 + 0.75e^t\right)^6$$

Determine the probability distribution of X. Find the mean and variance.

Solution

X ~ Bin (6, 0.75).

E(X) = 6*0.75 = 4.5. Var (X) = 6*0.75*0.25 = 1.125

Discrete Probability Distributions

3. **Poisson distribution**

Let X be a random variable denoting the number of successes in a sequence of n ($n \geq 30$) Bernoulli trials with probability p ($p < 0.5$) of successes, satisfying the conditions C1 – C3 of the binomial distribution such that $n\,p = \lambda$ is finite.

It is clear that X is a discrete r. v. with possible values X = 0, 1, 2, …

Discrete Probability Distributions

The probability mass function of X is

$$f(x) = P(X = x) = \frac{\lambda^x}{x!}e^{-\lambda}, x = 0,1,2,..., 0 < \lambda < \infty$$

In this case, we say X has a Poisson Distribution with parameter λ and write

$$X \sim Poi(\lambda)$$

Discrete Probability Distributions

Characteristics of Poisson distribution

(a) Poisson distribution is a limiting distribution of the binomial distribution as n tends to ∞.

(b) Poisson distribution is called the distribution of rare events. It is used when n is large and p is small, such that $\lambda = n\,p < \infty$.

(c) Mean = Variance = λ

(d) Standard deviation = $\sqrt{\lambda}$

Discrete Probability Distributions

(e) The MGF is

$$M_X(t) = e^{\lambda(\exp t - 1)}$$

(f) The PGF is

$$G_X(t) = e^{\lambda(t-1)}$$

Discrete Probability Distributions

Example (5)

The number X of annual earthquakes in a certain country has a mean 4. What is the probability distribution of X.

Solution

Because of the earthquake is a rare event, then the distribution of X is Poisson distribution with parameter $\lambda = 4$.

Discrete Probability Distributions

Example (6)

Consider the case when X has a Poisson distribution with parameter 3.

In this case:

(i) The PMF of X is $f(x) = P(X = x) = \frac{3^x}{x!} e^{-3}, x = 0,1,2,\ldots\infty$

(ii) E(X) = Var(X) = 3

(iii) P(X = 2) = f(2) = 4.5 e^{-3} = 0.224

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(iv) P(X is at least 3) = $P(X \geq 3) = f(3) + f(4) + \ldots$

$= 1 - [f(0) + f(1) + f(2)]$

$= 1 - 8.5\, e^{-3} = 1 - 0.423 = 0.577$

(v) P(X is at most 2) = $P(X \leq 2) = f(0) + f(1) + f(2)$

$= 8.5\, e^{-3} = 0.423$

Discrete Probability Distributions

4. **Geometric distribution**

Let X be a r. v. denoting the number of Bernoulli trials, required to obtain the first success.

The possible values of X are X = 1, 2, …

It is clear that X is a discrete r. v. with

$P(X = 1) = p$

$P(X = 2) = p\,q, \dots$

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In general $f(x) = P(X = x) = p\,q^{x-1},\; x = 1, 2, \ldots \infty$

This is called the geometric distribution with parameter p.

We denote this by writing $X \sim Geom(p)$

$$E(X) = \frac{1}{p}, \quad Var(X) \quad \frac{q}{p^2},$$

$$M_X(t) = \frac{p\,e^t}{1 - q\,e^t}, \quad G_X(t) = \frac{p\,t}{1 - q\,t}$$

Discrete Probability Distributions

Example (7)

Consider the case $X \sim Geom(0.65)$

This means that X has a geometric distribution with parameter p = 0.65

Thus

$$f(x) = P(X = x) = (0.65)(0.35)^{x-1}, x = 1, 2, \ldots \infty$$

We have:

Discrete Probability Distributions

(i) $E(X) = 1/0.65 = 1.538$

(ii) $Var(X) = 0.35 / (0.65)^2 = 0.828$

(iii) $P(X = 3) = f(3) = 0.080$

(iv) $P(X > 2) = 1 – f(1) – f(2) = 0.122$

(v) $P(X < 4) = f(1) + f(2) + f(3) = 0.957$

Discrete Probability Distributions

- **Example (8)** :

A person fires at a certain target in a sequence of independent successive trials with a constant probability 0.9 of hitting the target. Find the probability of hitting the target for the first time in:

(i) Exactly 2 trials

(ii) At least 4 trials

(iii) At most 3 trials

Discrete Probability Distributions

- **Solution**

  Let X = number of trials, required to hit the target for the first time. Consequently,

$$X \sim \text{Geo}\ (0.9).$$

$$f(x) = (0.9)(0.1)^{x-1}\ ,\ x = 1, 2, 3, \ldots$$

(i) Req. Prob. = $P(X = 2) = f(2) = \ldots$

(ii) Req. Prob. = $P(X \geq 4) = 1 - [f(1)+f(2)+f(3)] = \ldots$

(iii) Req. Prob. = $P(X \leq 3) = f(1) + f(2) + f(3) = \ldots$

Discrete Probability Distributions

- **Example (9)** :

  Find the mean, the variance, and the standard deviation of the number of hitting in **Example (8)**.

- Solution

Mean = $E(X) = 1/p = 1/0.9 = 1.111$

Variance = $Var (X) = q/p^2 = 0.1 / (0.9)^2 = 0.123$

Standard deviation = $\sigma = 0.351$

Discrete Probability Distributions

- **Example (10)** :

  The MGF of the random variable X is

$$M_X(t) = \frac{0.65e^t}{1-0.35e^t}$$

- Find the probability distribution of X. Find the mean, the variance, and the standard deviation of X.

Discrete Probability Distributions

- **Solution**

It is clear that X has a geometric distribution with parameter $p = 0.65$. Therefore

$$f(x) = P(X = x) = (0.65)(0.35)^{x-1}, \quad x = 1, 2, \ldots$$

Mean = $E(X) = 1/0.65 = 1.538$

Variance = $Var(X) = 0.35 / 0.65 = 0.828$

Standard deviation = $\sigma = 0.91$

Discrete Probability Distributions

5. **Negative binomial distribution**

Let X be a r. v. denoting the number of Bernoulli trials required to obtain the first k successes.

The possible values of X are k, k+1, k+2, …

It is clear that X is a discrete random variable.

We can prove that the PMF of X has the form:

$$f(x) = P(X = x) = {}^{x-1}C_{k-1} p^{k} q^{x-k}, x = k, k+1, \ldots \infty$$

Discrete Probability Distributions

In this case, we say that X has a negative binomial distribution with parameters k, p .

This is referred to by writing $X \sim NB(k, p)$

<u>Note that</u>

The negative binomial distribution reduces to the geometric distribution when k = 1.

Thus, the geometric distribution is a special case of the negative binomial distribution.

Discrete Probability Distributions

Characteristics of negative binomial distribution

(a) $E(X) = k/p$

(b) $Var(X) = k\,q/p^2$

(c) The MGF is $$M_X(\ ) = \left(\frac{p\,e^t}{1-q\,e^t}\right)^k$$

(d) The probability generating function is

$$G_X(t) = \left(\frac{p\,t}{1-q\,t}\right)^k$$

Discrete Probability Distributions

Example (11)

Consider the case when $X \sim NB(5, 0.8)$. This means that X has negative binomial distribution with k = 5 and p = 0.8.

In this case

(a) E(X) = 5/0.8 = 6.25

(b) Var (X) = 5*0.2 / $(0.8)^2$ = 1.563

(c)

$$P(X = 7) = \binom{6}{4}(0.8)^5(0.2) = 0.197$$

Discrete Probability Distributions

6. **Hyper-geometric distribution**

Consider a collection of k of objects of a certain Type and N – k of another Type.

A random sample of size n is drawn without replacement.

Let X be a random variable denoting the number of objects of the first type in the selected sample.

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The PMF of X is

$$f(x) = P(X = x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}, \quad x = \max(0, n+k-N), \ldots, \min(k, n)$$

In this case, we say that X has a hyper-geometric distribution with parameters N, k, n and write

$$X \sim HG(N, k, n)$$

Discrete Probability Distributions

Characteristics of hyper-geometric distribution

$$X \sim HG\,(N, k, n)$$

(i) Mean

$$E(X) = np = n\left(\frac{k}{N}\right)$$

(ii) Variance

$$Var(X) = npq = n\left(\frac{k}{N}\right)\left(1-\frac{k}{N}\right)\left(\frac{N-n}{N-1}\right)$$

Discrete Probability Distributions

<u>Example</u> (12)

From a group of 6 men and 4 women, a random sample of size 5 persons is selected without replacement. Let X denotes the number of men in the sample.

It is clear that X has HG(10, 6, 5). The PMF of X is

$$f(x) = P(X = x) = \frac{\binom{6}{x}\binom{4}{5-x}}{\binom{10}{5}}, \quad x = 1, \ldots, 5$$

Discrete Probability Distributions

We have

(i) P(2 men are selected) = P(X=2) = f(2) = 0.238

(ii) P(selecting 2 women) = P(X=3) = f(3) = 0.476

(iii) P(less than 3 men) = f(1)+f(2)

= 0.024 + 0.238

= 0.262

Discrete Probability Distributions

- **Example (13)** :

  Calculate the mean, variance, and standard deviation of X in Example (12).

  Solution

$$E(X) = np = n\left(\frac{k}{N}\right) = 3$$

$$Var(X) = npq = n\left(\frac{k}{N}\right)\left(1 - \frac{k}{N}\right)\left(\frac{N-n}{N-1}\right) = 0.667$$

Standard deviation = σ = 0.816