King Abdulaziz University
Faculty of Science
Department of Statistics

STAT 211
PROBABILITY THEORY (1)

STATISTICS & MATHEMATICS STUDENTS

CHAPTER 1
INTRODUCTION TO PROBABILITY

Introduction to Probability

- **Random Experiment**:

  It is an experiment , whose possible outcomes are known, but cannot be predicted with certainty.

- **Examples**:

  Example 1. Tossing a coin once

  Example 2. Rolling a die once

  Example 3. Drawing a card from a deck

Introduction to Probability

- **Sample Space**:

  It is a set whose elements represent all possible outcomes of a random experiment. It is usually denoted by S.

- **Examples** :

In Example (1): S = {Head, Tail} = {H, T}

In Example (2): S = {1, 2, 3, 4, 5, 6}

In Example (3): S = {1, 2, …, 10, Jack, Queen, King}

Introduction to Probability

- **More Examples** :

Example (4): Tossing a coin twice

S = {HH, HT, TH, TT}.

Example (5): Rolling a die twice

S = {(a, b) | a, b = 1, 2, …6}.

Example (6): Tossing a coin 3 times

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}.

Introduction to Probability

- **Example** (7): When tossing a coin until a Head appears, the sample space is

$$S = \{H, TH, TTH, \ldots\}$$

- **Example** (8): Random selection of a point on a line results in the sample space

$$S = \{x \mid -\infty < x < \infty\}$$

- **Example** (9): Random selection of a point in the x, y plane results in the sample space

Introduction to Probability

$$S = \{(x, y) \mid -\infty < x < \infty \text{ and } -\infty < y < \infty\}$$

- **Example** (10): Random selection of a point inside the unit circle results in the sample space $S = \{(x, y) \mid x^2 + y^2 < 1\}$
- **Example** (11): Random selection of a point outside the unit circle results in the sample space

$$S = \{(x, y) \mid x^2 + y^2 > 1\}$$

Introduction to Probability

- Sample spaces are classified into:

(i) **Finite** sample spaces: Examples 1 – 6.

(ii) **Denumerable** (countable) S. S.: Example 7

(iii) **Infinite** S. S.: Examples 8 – 11.

Introduction to Probability

- **<u>Event</u>** :

  It is a subset of the sample space. It is usually denoted by A, B, … .

- **<u>Examples</u>**:

<u>Example</u> (12): when rolling a die once

A = outcome is an even number = {2, 4, 6}

B = outcome is an odd number = {1, 3, 5}

C = outcome is divisible by 3 = {3, 6}

Introduction to Probability

Example (13): when rolling a die twice

A = getting a sum of 7

= {(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)}

B = getting a sum of at least 9

= {(3, 6), (4, 5), (5, 4), (6, 3), (4, 6), (5, 5), (6, 4), (5, 6), (6, 5), (6, 6)}

Introduction to Probability

C = getting a sum of at most 5

= {(1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1),

(1, 4), (2, 3), (3, 2), (4, 1)}

Introduction to Probability

- **Example** (14): When tossing a coin 3 times

A = getting one Head = {HTT, THT, TTH}

B = getting 2 Heads = {HHT, HTH, THH}

C = getting at least 1 Head

= {HTT, THT, TTH, HHT, HTH, THH, HHH}

D = getting at most 2 Heads

= {TTT, HTT, THT, TTH, HHT, HTH, THH}

Introduction to Probability

Logical Dictionary

Symbol	Set Theory	Probability Theory
A	set	event
A^c	Complement of A	Event “not A”
ϕ	Empty set	Impossible event
S	Universal set	Sample space
A U B	Union of A and B	Event “A or B”
$A \cap B$	Intersection of A and B	Event “Both A and B”
A – B	Difference between A and B	Event “A, but not B”
$A \cap B = \phi$	A and B Disjoint sets	A and B Mutually exclusive

Introduction to Probability

- **<u>Definition of Probability of an event</u>**

(1) <u>Subjective Approach</u>

It depends on the experience and the amount of available information

(2) **<u>Empirical Approach</u>**

It depends on repeating the experiment “n” times and noting the number “m” of occurrence of the event A, and taking

$P(A) = m / n$ for sufficiently large n.

Introduction to Probability

(3) **Classical Approach**

It depends on assuming that all outcomes are equally likely. In this case

$$P(A) = \frac{n(A}{n(S)}$$

where

n(A) = number of sample points in A

n(S) = number of sample points in S

Introduction to Probability

Mathematical (Axiomatic) Definition

It depends on Axioms of Probability:

(i) **Axiom 1**: For every event A in S: $P(A) \geq 0$

(ii) **Axiom 2**: $P(S) = 1$

(iii) **Axiom 3**: For mutually exclusive events A and B in S:

$$P(A \cup B) = P(A) + P(B)$$

Introduction to Probability

Some Basic Theorems

(1) For any event A in S:

$$0 \leq P(A) \leq 1$$

(2) For any event A in S:

$$P(A) + P(A^c) = 1$$

(3) For any events A, B in S:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Introduction to Probability

(4) For any events A , B in S:

$$P(A - B) = P(A) - P(A \cap B)$$

Special case:

when $B \subset A$, we have

$$P(A - B) = P(A) - P(B)$$

Introduction to Probability

(5) For events A, B such that $A \subset B$:

$$P(A) \leq P(B)$$

- Special case:

  When $A_1 \leq A_2 \leq \ldots \leq A_n$ we have

$$P(A_1) \leq P(A_2) \leq \ldots \leq P(A_n)$$

Introduction to Probability

(6) For any events A, B, C in S:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Introduction to Probability

Remark:

For mutually exclusive events A, B, and C in S:

(1) $P(A \cup B) = P(A) + P(B)$

(2) $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

Introduction to Probability

<u>Examples Based on Classical Definition</u>

- **<u>Example</u>** (15): When tossing a coin once, the sample space is

$$S = \{head , tail\} = \{H , T\}$$

P(Head) = ½

P(Tail) = ½

Introduction to Probability

- **Example** (16): When tossing a coin twice, the sample space is

$$S = \{HH , HT , TH , TT\}$$

P(2 Heads) = ¼,

P(2 Tails) = ¼,

P(1 Head) = ½

P(at least 1 Head) = ¾

Introduction to Probability

- **Example** (17): When tossing a coin 3 times, the sample space is

  S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

  P(1 Head) = 3/8,

  P(2 Heads) = 3/8,

  P(3 Heads) = 1/8,

  P(no Heads) = 1/8

Introduction to Probability

- **Example** (18): When rolling a die once

$$S = \{1, 2, 3, 4, 5, 6\}$$

P(face 1) = … = P(face 6) = 1/6 =0.167

P(even number) = 3/6 = 0.5,

P(odd number) = 3/6 = 0.5,

P(a number divisible by 3) = 2/6 = 0.333,

P(a prime number) = 3/6 = 0.5

Introduction to Probability

- **Example** (19): Rolling two dice once

$$S = \{(x, y) \mid x, y = 1, 2, 3, 4, 5, 6\}$$

P(odd number on first die) = 18/36 = 0.5

P(odd number on second die) = 18/36 = 0.5

P(odd number on both dice) = 9/36 = 0.25

P(getting a sum of 7) = 6/36 = 0.167

P(equal numbers on both dice) = 6/36 = 0.167

Introduction to Probability

- **Example** (20): Let A and B be defined on the same sample space S such that:

$$P(A) = 0.3, P(B) = 0.25, P(A \cap B) = 0.07$$

- P(A or B) = P(A U B) = P(A) + P(B) – P(A∩ B)

$$= 0.3 + 0.25 - 0.07$$

$$= 0.48$$

Introduction to Probability

Example (21): Let A, B, C be defined on the same sample space S such that

$$P(A) = 0.3, P(B) = 0.25, P(C) = 0.40, P(A \cap B) = 0.07,$$

$$P(A \cap C) = 0.09, P(B \cap C) = 0.08, P(A \cap B \cap C) = 0.03$$

P(at least 1 event will occur) = P(A U B U C)

$= P(A) + P(B) + P(C) - P(A\cap B) - P(A\cap C) - P(B\cap C) + P(A\cap B\cap C) = 0.74$

Introduction to Probability

P(only 1 event will occur)

$= P(A \cap B^c \cap C^c) + P(A^c \cap B \cap C^c) + P(A^c \cap B^c \cap C)$

$= 0.17 + 0.13 + 0.26$

$= 0.56$

P(only 2 events will occur)

$= P(A \cap B \cap C^c) + P(A \cap B^c \cap C) + P(A^c \cap B \cap C)$

$= 0.15$

Introduction to Probability

P(at least 2 events will occur)

$= P(A \cap B \cap C^c) + P(A \cap B^c \cap C) + P(A^c \cap B \cap C) +$

$P(A \cap B \cap C)$

$= 0.18$

P(no event will occur) $= P[(A \cup B \cup C)^c]$

$= 1 – P(A \cup B \cup C)$

$= 1 – 0.74$

$= 0.26$

Introduction to Probability

$$P[A \cap (B \cup C)] = P[(A \cap B) \cup (A \cap C)]$$
$$= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$$

= 0.07 + 0.09 – 0.03 = 0.13

$$P[A \cup (B \cap C)] = P(A) + P(B \cap C) - P(A \cap B \cap C)$$

= 0.3 + 0.08 – 0.03 = 0.35

Introduction to Probability

- **Permutations**:
- A permutation of n different objects is an arrangement of these n objects
- The number of permutations of n different objects, taken all at a time, is n!
- The number of permutations of n different objects, taken r (r ≤ n) at a time, is denoted by

$$^{n}P_{r} = \frac{n!}{(n-r)!}, 0 \le r \le n$$

Introduction to Probability

- **Remarks**:

  (i) ${}^{n}P_{0} = 1$

  (ii) ${}^{n}P_{n} = n!$

  (iii) ${}^{n}P_{1} = n$

  (iv) ${}^{n}P_{r} = n\,(n - 1)\,(n - 2) \ldots (n - r + 1)$

Introduction to Probability

- **Example** (22):

  By rearrangement of the letters of the word "PETROLIUM ", the number of

- 2-letter words = $^{9}P_{2} = 9 \times 8 = 72$
- 3-letter words = $^{9}P_{3} = 9 \times 8 \times 7 = 504$
- 4-letter words = $^{9}P_{4} = 9 \times 8 \times 7 \times 6 = 3024$
- 5-letter words = $^{9}P_{5} = 9 \times 8 \times 7 \times 6 \times 5 = 15120$

Introduction to Probability

- 6-letter words = ${}^{9}P_{6} = 9 \times 8 \times 7 \times 6 \times 5 \times 4$

  $= 60480$

- 7-letter words = ${}^{9}P_{7} = 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3$

  $= 181440$

- 8-letter words = ${}^{9}P_{8}$

  $= 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 362880$

- 9 – letter words = 9! = 362880

Introduction to Probability

- **Combinations :**

  The number of groups of r objects, selected at random from n objects is denoted and defined by

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!(n-r)!}, r \le n$$

Introduction to Probability

- **Remarks**:

(i) ${}^{n}C_{0} = 1$

(ii) ${}^{n}C_{1} = n$

(iii) ${}^{n}C_{n} = 1$

(iv) ${}^{n}C_{r} = {}^{n}C_{n-r}$

(v) ${}^{n}C_{r} = {}^{n}P_{r} / r!$

Introduction to Probability

- **<u>Example (23)</u>:**

  When tossing a coin 4 times, the number of
- outcomes with no Heads = $^4C_0 = 1$
- outcomes with 1Head = $^4C_1 = 4$
- outcomes with 2 Heads = $^4C_2 = 6$
- outcomes with 3 Heads = $^4C_3 = 4$
- outcomes with 4 Heads = $^4C_4 = 1$

Introduction to Probability

- **Basic Counting Principle**:

If a process P_1 can occur in n_1 different ways
and to each of these ways
a process P_2 can occur in n_2 different ways,
then
P_1 and P_2 will occur in $n_1 \times n_2$ different ways

Introduction to Probability

- **Generalized Counting Principle**

If a process P_1 can occur in n_1 different ways and to each of these ways a process P_2 can occur in n_2 different ways, and so on ..., then

Processes P_1 , P_2 , ..., and P_k will occur in

$$n_1 \times n_2 \times \ldots \times n_k$$

different ways

Introduction to Probability

- **Example (24)**:

  Five persons are selected at random from a group of 6 men and 8 women. Find the number of selections with 3 men & 2 women.

- **Solution**:

  Required Number = ${}^{6}C_{3} \times {}^{8}C_{2} = 20 \times 28 = 560$

Introduction to Probability

- **Example (25)**:

  Seven persons are selected at random from a group of 6 men, 8 women, and 4 children. Find the number of selections with 3 men, 2 women, and 2 children.

- **Solution**:

  Required Number = ${}^{6}C_{3} \times {}^{8}C_{2} \times {}^{4}C_{2}$

  $= 20 \times 28 \times 6 = 3360$

Introduction to Probability

- **Example (26)**:

  Four persons are selected at random from a group of 6 men, 8 women, and 4 children. Find the number of selections with equal number of men and women

- **Solution**:

  Req. = $^6C_0 \times {}^8C_0 \times {}^4C_4 + {}^6C_1 \times {}^8C_1 \times {}^4C_2 +$
  $+ {}^6C_2 \times {}^8C_2 \times {}^4C_0 = 709$

Introduction to Probability

- **Example (27)**:

  Five persons are selected at random from a group of 6 men, 8 women, and 4 children. Find the number of selections with 2 men and at least 1 woman.

- **Solution**:

  Required = ${}^{6}C_{2} \times {}^{8}C_{1} \times {}^{4}C_{2} + {}^{6}C_{2} \times {}^{8}C_{2} \times {}^{4}C_{1}$

  $+ {}^{6}C_{2} \times {}^{8}C_{3} \times {}^{4}C_{0} = 3240$

Introduction to Probability

- **Example (28)**:

Four persons are selected at random from a group of 6 men, 8 women, and 4 children. Find the number of selections in which all persons have the same gender

- **Solution**:

Required Number = ${}^{6}C_{4} + {}^{8}C_{4} + {}^{4}C_{4}$

$= 86$

Introduction to Probability

- **Example (29)**:

  Four persons are selected at random from a group of 6 men, 8 women, and 4 children. Find the number of selections in which all genders are not of the same gender.

- **Solution**:

  Required = N(S) – N(all of the same gender)

  $$= {}^{18}C_4 - [\,{}^{6}C_4 + {}^{8}C_4 + {}^{4}C_4\,]$$

  $$= 3060 - 86 = 2974$$

Introduction to Probability

Binomial Expansion

(n positive integer)

$$(x+y)^n = \sum_{k=0}^{n} {}^nC_k\, y^k x^{n-k} = {}^nC_0 x^n y^0 + {}^nC_1 x^{n\ 1} y^1 + \ldots + {}^nC_n x^0 y^n$$

$$(x+y)^n = x^n + n\, x^{n-1} y^1 + \frac{n(n-1)}{2!} x^{n-2} y^2 + \ldots + y^n$$

Introduction to Probability

- The coefficients in the binomial expansion can be determined from Pascal’s triangle:

		1	1		
	1	2	1		
	1	3	3	1	
1	4	6	4	1	
1	5	10	10	5	1

Introduction to Probability

- **<u>Example (30):</u>**

  In the expansion of $(x + y)^4$ the coefficients are respectively: 1, 4, 6, 4, 1.

$$(x+y)^4 = x^4 + 4x^3y^1 + 6x^2y^2 + 4x\,y^3 + y^4$$

- **<u>Example (31)</u>** :

  In the expansion of $(x + y)^5$ the coefficients are respectively: 1, 5, 10, 10, 5, 1.

- $$(x+y)^5 = x^5 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5x\,y^4 + y^5$$

Introduction to Probability

- **Multinomial coefficients**:

  The number of permutations of n objects that contains k_1 of type 1, k_2 of type 2, k_r of type r is denoted and defined by:

$$\binom{n}{k_1, k_2, \ldots, k_r} = \frac{n!}{k_1!\, k_2! \ldots k_r!}, \quad k_1 + k_2 + \ldots + k_r = n$$

Introduction to Probability

- **<u>Example (32)</u>**:

  Determine the number of words that can be formed by rearranging the letters of the word "STATISTICS".

  <u>Solution</u>. We have 10 letters containing : 3 "S", 3 "T", 2 "I", 1 "A", and 1 "C".

  Required Number = $10! / [3! \times 3! \times 2! \times 1! \times 1!]$

  $= 50400$